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Wave Equation of Symmetry Constrained Dirac Particles[1]

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Starting from the energy–momentum vector relationship for two observers, a multivector wave equation, which we call the full Dirac equation, is derived. It is shown to correspond to a series of symmetry constrained Dirac Fields with basic, repetitive, $SU(5)$ structure. The consequences of the symmetry restrictions for the definition of charge, hypercharge, color, charm, etc. are discussed together with a description of the associated gauge fields.

1. INTRODUCTION

An entirely new procedure of deriving the Dirac wave equation is presented here. It allows the generalization of the spin-1/2 field equations in a way more suitable to understand the physical meaning of the different terms in the (multivector) wave function and the behavior under the basic symmetries of space-time. In a previous work we developed this equation in matrix form (Keller, 1981).

The new wave equation, which we will call the full Dirac equation, can be studied, in particular, under the operation of duality rotation. Duality rotations mix space-time vectors with space-time three-vectors as they are dual to each other. The duality rotation D_α^μ can be performed on each of the four space-time vectors independently; the wave function can be single valued, multivalued, or invariant upon each of the D_α^μ. This will be on the origin of symmetry restrictions to the possible solutions to the full Dirac equation. Because the gauged Dirac equation, with the electromagnetic field as a gauge field, requires the duality rotation to be global (on all four

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space-time vectors simultaneously) only those symmetry constrained Dirac fields which have the same valuedness upon all four D^{μ}_{α} can interact with the electromagnetic field according to standard electrodynamics.

The elementary matter fields described by the full Dirac equations are given the generic name "symmetry *con*strained *Dirac* fields," or *diracons* for short.

In Section 2 the full Dirac (fD) equation is derived from the energy-momentum space-time vector. In Section 3 the duality rotation symmetries of the fD equation are studied and the resulting diracons are classified. The basic collection of gauge fields and their implications are presented. Section 4 gives some comment on the implications of this work in the analysis of quantum mechanics.

2. THE FULL DIRAC EQUATION

Consider the tangent space $\mathcal{D}(x)$ at point x of space-time consisting of the vectors $\{\gamma_\mu; \mu = 0, 1, 2, 3\}$. These orthonormal vectors

$$\gamma_\mu = \Box x^\mu, \qquad \gamma_0^2 = 1, \qquad \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1 \tag{1}$$

have their metric given by the symmetric product ($\cdot$ product)

$$g_{\mu\nu} = \gamma_\mu \cdot \gamma_\nu \equiv \tfrac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) \tag{2}$$

and can be used to define multivectors from their antisymmetric product ($\wedge$ product)

$$\gamma_{\mu\nu\cdots\lambda} \equiv \gamma_\mu \wedge \gamma_\nu \wedge \cdots \wedge \gamma_\lambda \tag{3}$$

defined by recursion, if b is an n-vector and a is a 1-vector

$$a \wedge b \equiv \tfrac{1}{2}\left[ab + (-1)^n ba\right] = a \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_n \tag{4}$$

Following Hestenes (1966) we define their total or geometrical product

$$ab = a \cdot b + a \wedge b \tag{5}$$

The observer $\mathcal{S}_1$ at x has at his disposal the 16 different basic multivectors or d numbers ($d =$ Dirac), including the scalar 1 and the pseudoscalar $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$, which constitute the Clifford algebra at $\mathcal{D}(x)$.

A Lorentz transformation

$$\mathcal{D}(x) \rightarrow \mathcal{D}'(x) = R(x)\mathcal{D}(x)R^{-1}(x) \tag{6}$$

with R being the sum of a scalar, a bivector, and a pseudoscalar such that

$$RR^{-1} = R^{-1}R = 1 \tag{7}$$

and

$$R^{-1} = \pm \tilde{R} \tag{8}$$

($\tilde{R}$ is a d number in which the order of all products has been reversed) can be written in terms of generators

$$R = \pm e^{-B/2} \tag{9}$$

the B being bivectors required to satisfy (8). The vectors associated with a second observer $\mathbb{S}_2$

$$\alpha_\mu = R\gamma_\mu R^{-1} \tag{10}$$

will again be orthogonal but not necessarily constant.

Consider now the energy momentum vector relation $\mathbf{p} = \mathbf{p}'$ for a "particle" moving with respect to observer $\mathbb{S}_1$

$$p^0\gamma_0 + p^1\gamma_1 + p^2\gamma_2 + p^3\gamma_3 = p'^0\gamma'_0 + p'^1\gamma'_1 + p'^2\gamma'_2 + p'^3\gamma'_3 \tag{11}$$

the primed components are those computed by $\mathbb{S}_1$ when the particle's movement is referred to a $\mathbb{S}'_1$ inertial system.

Here

$$\gamma'_\alpha = R\gamma_\alpha\tilde{R} \tag{12}$$

and

$$\gamma_\alpha = \tilde{R}\gamma'_\alpha R \tag{13}$$

then

$$p^\alpha\gamma_\alpha = p'^\alpha R\gamma_\alpha\tilde{R} \tag{14}$$

on multiplying by R on the right

$$p^\alpha\gamma_\alpha R = p'^\alpha R\gamma_\alpha \tag{15}$$

the explicit dependence on $\mathbb{S}'_1$ has disappeared, such that

$$\left(p^\beta\gamma_\beta R\right)\cdot\left(\tilde{R}p^\alpha\gamma_\alpha\right) = \left(p'^\beta R\gamma_\beta\right)\cdot\left(p'^\alpha\gamma_\alpha\tilde{R}\right) \tag{16}$$

using the transpose to equation (15)

$$\tilde{R}p^{\alpha}\gamma_{\alpha}=p'^{\alpha}\gamma_{\alpha}\tilde{R} \tag{17}$$

to obtain the standard scalar invariant

$$p^{\beta}p^{\alpha}g_{\beta\alpha}=p'^{\beta}p'^{\alpha}g_{\beta\alpha} \tag{18}$$

We may then make a further Lorentz transformation with the even multivector

$$Q=\exp[-I(\mathbf{p}\cdot\mathbf{x}+\gamma_5\mathbf{p}'\cdot\mathbf{x}')/\hbar], \qquad I^2=-1 \tag{19}$$

with the particular (rotation plane) choice

$$I_i=\gamma_5\gamma_0\gamma_i, \qquad i=1,2,3 \tag{20}$$

defining

$$RQ=\psi \quad \text{and} \quad \tilde{\psi}=\tilde{Q}\tilde{R} \tag{21}$$

to obtain the eigenvalue equation

$$\gamma_{\alpha}g_{\alpha\alpha}\partial_{\alpha}\psi I_i=-g_{\alpha\alpha}\partial'_{\alpha}\psi\gamma_5 I_i\gamma_{\alpha} \tag{22}$$

The Q contain eight phase factors in total to change $R\to\psi$. This is the full Dirac equation in multivector form.

The $\gamma_5\gamma_{\mu}=\gamma_{\mu}^{D}$ are the dual of the γ_{μ} and linearly independent. ψ is now an even d number depending on $\{x^{\mu}\}$ and $\{x'^{\mu}\}$ and (22) may be properly called a quantum-mechanical-like particle's wave equation. The particular choice of $\sigma_3=\gamma_3\gamma_0$ in the phase factor reflects the fact that two currents can be defined which will be considered,

$$J_0=\tilde{\psi}\gamma_0\psi \quad \text{and} \quad J_3=\tilde{\psi}\gamma_3\psi \tag{23}$$

The γ_5 is needed to have both the bivector σ_3 and its dual to keep the two forms in the exponent of (18) linearly independent phase factors of ψ. The $g_{\mu\mu}$ appear in (21) in order to have the correct signs for the p^{μ} and p'^{μ} because time- and spacelike terms in $\mathbf{p}\cdot\mathbf{x}$ and $\mathbf{p}'\cdot\mathbf{x}'$ have opposite signs.

The choice of $I=I_3=\gamma_5\sigma_3=\gamma_1\gamma_2$ is arbitrary and $I=I_2=\gamma_5\sigma_2$ or $I=I_1=\gamma_5\sigma_1$ could as well be considered. The wave equation (18) is then $SU(2)$ symmetric and, because $I=-\tilde{I}$, it is also $U(1)$ symmetric. Then it is $SU(2)\times U(1)$ symmetric as basic gauge transformations symmetries.

With

$$\mathbf{p}' = m_0 \gamma_0' \tag{24}$$

then (22) reduces to

$$\gamma_\alpha g_{\alpha\alpha} \partial_\alpha \psi I_i = m_0 \psi \gamma_0 \tag{25}$$

This equation has been discussed in length by Hestenes (1966) which derived it in a different way (Hestenes, 1975). It has the same form as the one derived by Gürsey (1958), with ψ being a four-spinor, in his study of the nucleons.

The use of the I_1, I_2, and I_3 as a base for the $SU(2)\times U(1)$ and the relation with the electroweak interaction has been discussed by Hestenes (1982).

3. DUALITY ROTATION SYMMETRIES OF THE FULL DIRAC EQUATION

The gauge transformations admit other symmetries which leave (19) invariant, a duality transformation

$$\begin{aligned} \gamma_\mu &\to \gamma_\mu^K = \cos\theta_\mu \gamma_\mu + \sin\theta_\mu \gamma_\mu^D \\ \gamma_\mu^D &\to \gamma_\mu^{DK} = -\sin\theta_\mu \gamma_\mu + \cos\theta_\mu \gamma_\mu^D \end{aligned} \tag{26}$$

or in multivector notation

$$\gamma_\mu \to \gamma_\mu^K = e^{\gamma_5 \theta^\mu t/2} \gamma_\mu e^{-\gamma_5 \theta^\mu t/2} = e^{\gamma_5 \theta^\mu t} \gamma_\mu = \gamma_\mu e^{-\gamma_5 \theta^\mu t} \tag{27}$$

The second and third equalities hold because γ_5 anticommutes with all γ_μ and γ_μ^D. Then the duality rotation factors can be incorporated directly in the ψ. The dependence on duality rotations should be considered for the wave function and the quantities, like the field's stress-energy tensor, constructed from it. They should be duality rotation invariant. Elementary particles interacting with electromagnetic fields should have some special symmetry on duality transformations (26) as the electromagnetic field stress-energy tensor is known to be duality invariant, even if the field amplitudes do transform. For this reason, charged particles should be described by a wave function single or double valued with respect to the four duality rotations θ^μ going from 0 to 2π.

There are three spacelike gauge duality rotations, equivalent among themselves $t_i = 2a_i$, and one timelike $t_0 = 2b$. The first few combinations of single-valued $t = 0, 2, \ldots$ and double-valued $t = 1, 3, \ldots$, symmetries for ψ are given in Table I.

The reason for this can be made more explicit by doing the polar transformations

$$(\rho^\mu)^2 = (\chi^\mu)^2 + (\chi'^\mu)^2, \qquad \rho^\mu > 0 \tag{28a}$$

$$\theta^\mu = -\arctan(\chi'^\mu/\chi^\mu) \tag{28b}$$

$$\gamma_\mu^+ = \tfrac{1}{2}(\gamma_\mu + \gamma_5\gamma_\mu) \quad \text{and} \quad \gamma_\mu^- = \tfrac{1}{2}(\gamma_\mu - \gamma_5\gamma_\mu) \tag{29}$$

to obtain

$$\left\{\gamma_\mu^+ e^{-i\theta^\mu}\left[\partial_{\rho^\mu} - (\gamma_5/\rho^\mu)\partial_{\theta^\mu}\right] + \gamma^- e^{i\theta^\mu}\left[\partial_{\rho^\mu} + (\gamma_5/\rho^\mu)\partial_{\theta^\mu}\right]\right\}\Psi^K = 0 \tag{30}$$

with

$$\Psi^K = e^{\gamma_5 t_\mu \theta^\mu}\psi \tag{31}$$

which is of the same form as from the considerations following (28).

What can be the relations between the a_i, the b, and the particles of the different matter fields? a_1, a_2, a_3 suggest the three "colors" and b suggest

TABLE I. Simplest Combinations of Quantum Numbers $t/2$ Corresponding to Duality Rotation of the Wave Function ψ of a Symmetry Constrained Dirac particle (Diracon)

a_1	a_2	a_3	b	Σa	Possible identification of the corresponding matter field
0	0	0	1/2	0	Neutrino
±1/2	0	0	1/2	±1/2	Lepton ($b = 0$) three types,
0	±1/2	0	1/2	±1/2	probable components of mesons
0	0	±1/2	1/2	±1/2	and barions
±1/2	±1/2	0	1/2	±2/2	Lepton ($b = 0$) three types,
0	±1/2	±1/2	1/2	±2/2	probable component of mesons
±1/2	0	±1/2	1/2	±2/2	and barions
±1/2	±1/2	±1/2	1/2	±3/2	Electron
±1/2	±1/2	±1/2	1	±3/2	Neutrino ($b = 1$)
±1	±1/2	±1/2	1	±2	Lepton ($b = 1$) three types,
±1/2	±1	±1/2	1	±2	probable component of heavy meason s
±1/2	±1/2	±1	1	±2	and barions
±1	±1	±1/2	1	±2 1/2	Lepton ($b = 1$) three types,
±1	±1/2	±1	1	±2 1/2	probable component of heavy
±1/2	±1	±1	1	±2 1/2	mesons and barions
±1	±1	±1	1	±3	Muon

charm in the form: $b=1/2$, first family of leptons and building block of barions, $b=1$ to be the second family or the charmed family, and $b=n$ in general the nth family of particles. I suggest the different fields, which will be Dirac fields constrained by symmetry should take the generic name of diracons as they all obey (a generalized, multivector) Dirac equation.

The dynamics of the matter fields are:

1. symmetric in a_1, a_2, a_3, and b
2. a_1 and a_2 and a_3 invariant
3. a_i and a_j invariant, $i \neq j$
4. a_i invariant
5. symmetric in a_1, a_2, a_3 but not on b
6. a_i changing into a different value or into a a_j.

Field 1 may correspond to the normal electromagnetic interaction. Fields 2, 3, 4, and 5 correspond to symmetry restricted electromagnetic interactions of diracon fields (not to fractional charges as in the quark theory), 5 to the weak interaction with $\Delta a = \pm 1/2$, whereas 6 is just the gluon type of interaction changing one color particle into another.

The theory of diracons is more general than $SU(5)$ and contains the $SU(3)$ and $SU(2) \times U(1)$ theories of elementary particles.

From the corresponding gauge transformations the properties of the different force mediating boson fields are easily derived. The conditions to observe diracons with electromagnetic interactions are also derived in this way.

The possibility of a diracon's interacting with the electromagnetic and other gauge field appears as a symmetry condition. If the duality rotations phase factors can be factorized, the field particles will interact electromagnetically; if the duality rotations phase factor cannot be factorized then the field's particles cannot obey standard electrodynamics. In the case of a composite particle the total wave function will be a product of the constituent field ψ, their product may have the required combination of duality rotation phase factors, and the composite particle obey standard electrodynamics. Symmetry constrained electrodynamics is not to be ruled out for other diracon fields or combination of fields.

The opposite procedure can be considered, the symmetries of the interaction carrying fields being fixed and, as a consequence, the symmetries of the diracon fields (in particular under duality rotations) be obtained.

4. QUANTUM MECHANICS

The multivector equation (25) has been shown by Hestenes to correspond to the standard Dirac equation, once a matrix representation of the vectors γ_μ is selected and a Dirac four-spinor $\Psi = \psi u$ is defined. The matrix u is a column matrix with only the first element being different from zero.

The procedure described in equations (12)–(22) and the analysis preceding equation (31) gives a straightforward interpretation of the wave function (31). It is first clear that the Dirac theory is a way of relating the energy–momentum vector from two different reference frames using the basic set γ_μ of only one observer. The wave function contains the information of the second (particle at rest or proper) frame and of the action associated with the particle. Knowing this, the relations between the observer's frame and the particle's rest frame can be accounted for and the action phase factor found.

More interesting is the case of the gauged Dirac equation to account for interation carrying fields. In this case the energy–momentum vector depends on time and position, the particle timelike vector

$$\alpha_0^2 = \frac{1}{1 - v^2} \tag{32}$$

and the square of the wave function gets a time dilatation factor which increases in the regions where the particle increases its velocity (attractive potential). A new type of question may now be asked: What is the relative probability of finding a particle at two places (x_1, t_1) and (x_2, t_2)? The ratios of the absolute values of the timelike vectors (usually called ρ) obtained from (10) contain this, probabilistic, information which is the origin of the probabilistic interpretation of quantum mechanics.

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